LINEAR DEPENDENCE OF QUOTIENTS OF ANALYTIC FUNCTIONS OF SEVERAL VARIABLES WITH THE LEAST SUBCOLLECTION OF GENERALIZED WRONSKIANS

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Abstract. We study linear dependence in the case of quotients of analytic functions in several variables (real or complex). We identify the least subcollection of generalized Wronskians whose identical vanishing is sufficient for linear dependence. Our proof admits a straight-forward algebraic generalization and also constitutes an alternative proof of the previously known result that the identical vanishing of the whole collection of generalized Wronskians implies linear dependence. Motivated by the structure of this proof, we introduce a method for calculating the space of linear relations. We conclude with some reflections about this method that may be promising from a computational point of view.

1. Introduction

As is well-known, a finite set of analytic functions of a single variable is linearly dependent if and only if its Wronskian is (identically) zero. A generalization of this to multivariate polynomials was made by Roth [11] in the course of an application to number theory. Roth's work states that a finite set of multivariate polynomials is linearly dependent if and only if its generalized Wronskians are zero. This result is also true for a finite set of quotients of analytic functions, which may be seen by either extending an argument presented by Cassels [5] (pp. 112-113), using Wolsson's generalization of a theorem of Bôcher [14], or noting work of Berenstein, Chang, and Li [2].

Roth notes in a footnote of his work [11], that there may exist relations among generalized Wronskians and their derivatives, so that the identical vanishing of some may imply the identical vanishing of others. Part of this article will demonstrate the least subcollection of generalized Wronskians whose identical vanishing establishes linear dependence, in the context of quotients of analytic functions. As we will

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point out at the end of Section 3, the proofs here readily hold in a broader algebraic context.

In Section 2. we begin with the relevant definitions and some background for the study of linear dependence. In Section 3, we introduce *Young-like sets*, which are a higher-dimensional analog of Young diagrams. We prove that the identical vanishing of the subcollection of generalized Wronskians associated with Young-like sets is sufficient for, and hence equivalent to, linear dependence in the context of quotients of multivariate analytic functions. We will show that this result is sharp, in the sense that any subcollection of generalized Wronskians whose collective, identical vanishing implies linear dependence must contain the generalized Wronskians associated with Young-like sets. In Section 4, using some of the concepts from the proofs of Section 3 we introduce a related procedure for determining the space of linear relations and offer some observations that touch on some calculational considerations.

2. Definitions and Background

We will begin with some definitions and background associated with linear dependence in general, and then we will direct our focus onto the case of analytic functions and their quotients.

Let Ω be a (connected) domain in $\mathcal{K}^m$, where $\mathcal{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. We use coordinates $x_1, x_2, \ldots, x_m$ on $\mathcal{K}^m$. Let $f_1, f_2, \ldots, f_N$ be $\mathcal{K}$-valued functions defined on Ω. Let ϕ be the row vector $[f_1, f_2, \ldots, f_N]$. The functions $f_1, f_2, \ldots, f_N$ are linearly dependent over $\mathcal{K}$ if and only if there exists constants $c_1, c_2, \ldots, c_N$ in $\mathcal{K}$, not all zero, such that $\sum_j^N c_j f_j = 0$ on Ω, or equivalently, if there exists a non-zero column vector c in $\mathcal{K}^N$ such that $\phi c = 0$ on Ω.

Let $\mathcal{T} = \mathcal{T}_m$ be the set of multi-indices, that is the m-tuples of non-negative integers. Per usual multi-index notation, for $\alpha = (a_1, a_2, \ldots, a_m) \in \mathcal{T}$, let $|\alpha| = \sum_j^m a_j$, $x^\alpha = x_1^{a_1} x_2^{a_2} \cdots x_m^{a_m}$, and $D^\alpha = D_{x_1}^{a_1} D_{x_2}^{a_2} \cdots D_{x_m}^{a_m}$. Let e_j denote the multi-index with 1 in the jth spot, and 0 elsewhere. For $\alpha, \beta \in \mathcal{T}$, let $\alpha \preceq \beta$ denote that α lexicographically precedes or equals β, i.e. $\alpha = (a_1, a_2, \ldots, a_m) \preceq \beta = (b_1, b_2, \ldots, b_m)$ means that either $\alpha = \beta$ or there exists an j_0 such that $a_j = b_j$ for $j < j_0$ and $a_{j_0} < b_{j_0}$. Also define the partial ordering $\leq$ on $\mathcal{T}$, by saying that $(a_1, a_2, \ldots, a_m) \leq (b_1, b_2, \ldots, b_m)$ if and only if $a_j \leq b_j$ for all j.

For a finite list of multi-indices $\alpha^1, \alpha^2, \ldots, \alpha^k \in \mathcal{T}$, we define

$$(1) \qquad M^{\phi}_{\alpha^1, \alpha^2, \ldots, \alpha^k} = \begin{bmatrix} D^{\alpha^1}\phi \\ D^{\alpha^2}\phi \\ \vdots \\ D^{\alpha^k}\phi \end{bmatrix} = \begin{bmatrix} D^{\alpha^1} f_1 & D^{\alpha^1} f_2 & \cdots & D^{\alpha^1} f_N \\ D^{\alpha^2} f_1 & D^{\alpha^2} f_2 & \cdots & D^{\alpha^2} f_N \\ \vdots & \vdots & & \vdots \\ D^{\alpha^k} f_1 & D^{\alpha^k} f_2 & \cdots & D^{\alpha^k} f_N \end{bmatrix},$$

and let $W^{\phi}_{\alpha^1,\alpha^2,\ldots\alpha^N} = \det(M^{\phi}_{\alpha^1,\alpha^2,\ldots\alpha^N})$. Suppose that $A = \{\alpha^1, \alpha^2, \ldots, \alpha^k\} \subseteq \mathcal{T}$, where $\alpha^1, \alpha^2, \ldots \alpha^k$ are labeled in lexicographical order. Let M^{ϕ}_A denote $M^{\phi}_{\alpha^1,\alpha^2,\ldots,\alpha^k}$, and, if A has cardinality N, let $W^{\phi}_A = \det(M^{\phi}_A)$.

Let $\mathcal{T}(k) = \{\alpha \in \mathcal{T} \mid |\alpha| < k\}$ for any positive integer k. Let $\mathcal{G}$ be the collection of all subsets $A \subseteq \mathcal{T}$ such that A has cardinality N and $A \cap \mathcal{T}(j)$ has cardinality at least j for $1 \leq j \leq N$. (Alternatively, a set A of cardinality N is in $\mathcal{G}$ if its elements can be labeled as $\alpha^1, \alpha^2, \ldots .\alpha^N$, not necessarily in lexicographical order, such that $|\alpha^j| < j$ for $1 \leq j \leq N$.) Using the terminology introduced by Roth [11], the determinants W^{ϕ}_A, $A \in \mathcal{G}$ are called the *generalized Wronskians* of the functions contained in ϕ.

If we assume that the entries of ϕ are linear dependent over $\mathcal{K}$ and contained in $C^{N-1}(\Omega)$, then there exists a non-zero column vector c in $\mathcal{K}^N$ such that $\phi c = 0$, and it follows that $(D^{\alpha}\phi)c = D^{\alpha}(\phi c) = 0$ for all $\alpha \in \mathcal{T}(N)$. So the column vector c is a null vector for any matrix M^{ϕ}_A with $A \subseteq \mathcal{T}(N)$. As this shows, the identical vanishing of the generalized Wronskians is a necessary condition for linear dependence.

Expository Note: In the case of single variable functions, there exists only one generalized Wronskian, which is the classically defined Wronskian. For general (not necessarily analytic) functions of a single variable, assuming appropriate regularity for the Wronskian to be defined, the identical vanishing of the Wronskian is not sufficient for linear dependence. For a simple example, consider any collection of C^{∞} bump functions whose supports have empty intersection. A more classic example, attributed to Peano [9], is the pair of functions $f_1(x) = x^2$ and $f_2(x) = x|x|$. A fair amount of work has considered what conditions may be adjoined to the identical vanishing of the Wronskian to form a sufficient condition for linear dependence in the single-variable case. See Peano [10], Bôcher [4], Curtiss [6], Meisters [8], and Wolsson [13]. When one adds that the functions are analytic, then an identically vanishing Wronskian is sufficient for linear dependence [3]. Analogously, for general (not necessarily analytic) functions of several variables, with appropriate regularity assumed, the identical vanishing of the generalized Wronskians is not sufficient for linear dependence. Some examination of the general multivariate case has been given by Wolsson [14].

Now we turn our attention to analytic functions and quotients of analytic functions. In this setting, the matter of linear dependence on a domain can be reduced to linear dependence on any neighborhood of any point in the domain, due to the analytic continuation of relations.

Let $\mathcal{O}$ denote the ring of germs of analytic functions about the origin in $\mathcal{K}^m$. Let $\mathcal{M}$ denote the fraction field of $\mathcal{O}$ or the field of germs of quotients of analytic functions about the origin in $\mathcal{K}^m$. (In the case that $\mathcal{K} = \mathbb{C}$ then $\mathcal{M}$ is the field of germs of meromorphic functions about the origin in $\mathbb{C}^m$.) For the results that follow, a more algebraically general definition for $\mathcal{O}$ (and $\mathcal{M}$) could be used instead; see the remark at the end of Section 3.

Here is a formulation of Roth's theorem, generalized to our present setting.

Theorem 2.1. *Assume that $f_1, f_2, \ldots, f_N$ are contained in $\mathcal{M}$. The entries of $\phi = [f_1, f_2, \ldots, f_N]$ are linear dependent over $\mathcal{K}$ if and only if all generalized Wronskians W^{ϕ}_{A}, $A \in \mathcal{G}$, are zero.*

Historical Note: Roth introduced and established his condition in the case of multivariate polynomials [11]. An earlier result for multivariate polynomials was established by Siegel [12]. An even earlier statement was given for analytic functions by Kellogg [7], but without proof. Kellogg's condition can be succinctly expressed in our notation as saying that linear dependence of the entries of ϕ is equivalent to the matrix $M^{\phi}_{\mathcal{T}(N)}$ having rank less than N. Siegel's result additionally specifies that there is equality between the degree of linear dependence of the entries of ϕ and the degree to which $M^{\phi}_{\mathcal{T}(N)}$ has rank less than N.

Proofs of Theorem 2.1 can be derived from arguments or results already existing in the literature. For one, a concise, direct proof presented by Cassels [5] (pp.112-113) for the case of multivariate rational functions readily works in the case of quotients of analytic functions. Separately, a result by Wolsson (explicitly stated for $\mathcal{K} = \mathbb{R}$ but extendible to $\mathcal{K} = \mathbb{C}$) regarding the general (non-analytic) case [14] (Theorem 1) can be used to imply Theorem 2.1. In the specific case that $\mathcal{K} = \mathbb{C}$ and $f_1, f_2, \ldots, f_N$ are entire functions, Berenstein, Chang, and Li give a stronger result than that of Wolsson and explicitly derive Theorem 2.1 as a corollary [2] (Corollary 2.2). Their approach also holds for quotients of analytic functions (real or complex). Additionally, the content of Section 3 will provide an independent means of showing Theorem 2.1.

When a selected generalized Wronskian $W^{f_1,\ldots,f_{N-1}}_{\alpha^1,\ldots\alpha^{N-1}}$ of $f_1, \ldots, f_{N-1}$ is not zero, the identical vanishing of the particular generalized Wronskians $W^{f_1,\ldots,f_{N-1},f_N}_{\alpha^1,\ldots,\alpha^{N-1},\alpha^k+e_j}$ for $1 \leq k \leq N-1$ and $1 \leq j \leq m$ is sufficient to imply linear dependence of $f_1, \ldots, f_N$, as Berenstein, Chang, and Li explicitly demonstrate in the case that $\mathcal{K} = \mathbb{C}$ and the functions $f_1, f_2, \ldots, f_N$ are entire [2] (Theorem 2.1). Also, as they point out, their arguments hold in a broader context [2] (Remarks (3) and (4)). (However, Remark (4) needs to be carefully interpreted. For functions $f_1, \ldots, f_N \in C^{N-1}(\Omega)$, their theorem holds locally about points in $\Omega \backslash \mathcal{Z}$, where $\mathcal{Z} = \{x \in \Omega \mid W^{f_1,\ldots,f_{N-1}}_{\alpha^1,\ldots\alpha^{N-1}} = 0\}$, but it can fail locally about points in $\mathcal{Z}$. cf. the paper by Wolsson [14].) For analytic functions and their quotients, which is our present focus, their result holds locally about any point, due to the continuation of relations.

So the assumption that a specific generalized Wronskian of $f_1, \dots, f_{N-1}$ is not zero causes the identical vanishing of a proper subcollection of the generalized Wronskians of $f_1, f_2, \dots f_N$ to be sufficient to imply linear dependence of $f_1, f_2, \dots, f_N$. Even without added assumptions, there is a proper subcollection of the generalized Wronskians whose identical vanishing implies linear independence for quotients of analytic functions, as we will see in the next section.

3. The Least Collection of Generalized Wronskians

The central idea of Theorem 2.1 is that the identical vanishing of the generalized Wronskians proves to be sufficient for linear dependence in the case of germs of analytic functions. However this result can be improved by replacing the collection of generalized Wronskians with a smaller subcollection of generalized Wronskians. The particular subcollection that we introduce proves to be the least one that satisfies this. These statements are the content of Theorem 3.1 and Theorem 3.4

We call a set $A \subseteq \mathcal{T}$ *Young-like* if $\alpha \in A$, $\beta \in \mathcal{T}$, and $\beta \leq \alpha$ imply that $\beta \in A$. For $m = 2$, finite Young-like sets correspond to Young diagrams. (More generally, Young-like sets can be identified with m-dimensional partitions having entries bounded by 1.) Let $\mathcal{Y}$ be the collection of all Young-like sets in $\mathcal{T}$ of cardinality N. Note that $\mathcal{Y} \subseteq \mathcal{G}$.

Theorem 3.1. *Assume that $f_1, f_2, \dots, f_N$ are contained in $\mathcal{M}$. The entries of $\phi = [f_1, f_2, \dots, f_N]$ are linear dependent over $\mathcal{K}$ if and only if the generalized Wronskians W^ϕ_Y associated with Young-like sets, $Y \in \mathcal{Y}$, are zero.*

Remark: Theorem 3.1 implies Theorem 2.1.

For $\alpha \in \mathcal{T}$, we define $\phi_\alpha = D^\alpha(\phi)$, with differentiation performed entry-wise. We will regularly view a $M \times N$ matrix with entries in $\mathcal{M}$ as a $\mathcal{M}$-linear map on column vectors from $\mathcal{M}^N$ to $\mathcal{M}^M$. Thus a $1 \times N$ matrix, such as ϕ or ϕ_α, is considered to have its kernel reside in $\mathcal{M}^N$. For $A \subseteq \mathcal{T}$, let $\mathcal{N}_A = \mathcal{N}^\phi_A = \bigcap_{\alpha \in A} \ker \phi_\alpha \subseteq \mathcal{M}^N$, which is the null space of M^ϕ_A.

Lemma 3.2. *For a $1 \times N$ matrix ϕ with entries in $\mathcal{M}$, there exists a Young-like set $Y \subseteq \mathcal{T}$ with cardinality at most N such that $\mathcal{N}_Y = \mathcal{N}_\mathcal{T}$.*

Proof. Define $S_\alpha = \{\beta \in \mathcal{T} \mid \beta \prec \alpha\}$, and let $Y = Y(\phi) = \{\alpha \in \mathcal{T} \mid \mathcal{N}_{S_\alpha} \neq \mathcal{N}_{S_\alpha \cup \{\alpha\}}\} = \{\alpha \in \mathcal{T} \mid \mathcal{N}_{S_\alpha} \not\subseteq \ker \phi_\alpha\}$.

Suppose that β and γ are any multi-indices satisfying $\beta \leq \gamma$ and $\beta \notin Y$. Then $\mathcal{N}_{S_\beta} \subseteq \ker \phi_\beta$, and so ϕ_β is a $\mathcal{M}$-linear combination of $\{\phi_\alpha\}_{\alpha \in S_\beta}$. In other words,

$$\phi_\beta = \sum_{\alpha \in S_\beta} a_\alpha \phi_\alpha, \tag{2}$$

for coefficients $a_\alpha \in \mathcal{M}$, $\alpha \in S_\beta$, with only finitely many being non-zero. Let δ equal $\gamma - \beta$. Differentiating (2) by D^δ, yields ϕ_γ on the left and a $\mathcal{M}$-linear combination of ϕ_α, $\alpha \in S_\gamma$ on the right. Thus $\mathcal{N}_{S_\gamma} \subseteq \ker \phi_\gamma$, implying $\gamma \notin Y$. Therefore Y is Young-like.

We claim that $\mathcal{N}_Y \subseteq \mathcal{N}_{S_\alpha \cup \{\alpha\}}$ for all $\alpha \in \mathcal{T}$. Since $\mathcal{T}$ is well-ordered under $\preceq$, we may establish this claim using transfinite induction. Consider any multi-index β where $\mathcal{N}_Y \subseteq \mathcal{N}_{S_\alpha \cup \{\alpha\}}$ for all $\alpha \in S_\beta$, yielding that $\mathcal{N}_Y \subseteq \bigcap_{\alpha \prec \beta} \mathcal{N}_{S_\alpha \cup \{\alpha\}} = \mathcal{N}_{S_\beta}$. If $\beta \notin Y$, then $\mathcal{N}_Y \subseteq \mathcal{N}_{S_\beta} = \mathcal{N}_{S_\beta \cup \{\beta\}}$. If $\beta \in Y$, then $\mathcal{N}_Y \subseteq \ker \phi_\beta$ which implies that $\mathcal{N}_Y \subseteq \mathcal{N}_{S_\beta} \cap \ker \phi_\beta = \mathcal{N}_{S_\beta \cup \{\beta\}}$. Therefore, by induction, the claim holds, so $\mathcal{N}_Y \subseteq \bigcap_{\alpha \in \mathcal{T}} \mathcal{N}_{S_\alpha \cup \{\alpha\}} = \mathcal{N}_{\mathcal{T}}$. Since the inclusion $\mathcal{N}_{\mathcal{T}} \subseteq \mathcal{N}_Y$ is clear, the equality $\mathcal{N}_Y = \mathcal{N}_{\mathcal{T}}$ follows.

It only remains to show that Y has cardinality at most N. Suppose, for sake of contradiction, that Y contains $N+1$ distinct elements, $\alpha^1 \prec \alpha^2 \prec \ldots \prec \alpha^{N+1}$. Then

$$\mathcal{M}^N \supsetneq \bigcap_{\alpha \preceq \alpha^1} \ker \phi_\alpha \supsetneq \bigcap_{\alpha \preceq \alpha^2} \ker \phi_\alpha \supsetneq \ldots \supsetneq \bigcap_{\alpha \preceq \alpha^{N+1}} \ker \phi_\alpha \tag{3}$$

is a strictly decreasing sequence of vector spaces, the last of which has codimension at least $N+1$ within $\mathcal{M}^N$, which yields a contradiction. □

Lemma 3.3. *For a $1 \times N$ matrix ϕ with entries in $\mathcal{M}$,*

$$\mathcal{N}_{\mathcal{T}} = \left(\mathcal{N}_{\mathcal{T}} \cap \mathcal{K}^N\right) \otimes_{\mathcal{K}} \mathcal{M}. \tag{4}$$

Proof. The right-hand side of (4) is tautologically contained in $\mathcal{N}_{\mathcal{T}}$. For the reverse inclusion it suffices to show that there exists a $\mathcal{M}$-basis for $\mathcal{N}_{\mathcal{T}}$ contained in $\mathcal{N}_{\mathcal{T}} \cap \mathcal{K}^N$.

Let $v_1, v_2, \ldots, v_k$ be a reduced $\mathcal{M}$-basis for $\mathcal{N}_{\mathcal{T}}$. Specifically we mean that there exist distinct indices $\ell_1, \ell_2, \ldots, \ell_k$ such that $(v_i)_{\ell_j} = \delta_{i,j}$, where $(v)_\ell$ denotes the ℓth entry of v and $\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

Observe that $D_{x_n} v_i \in \mathcal{N}_{\mathcal{T}}$ for any i and n, since

$$D^\alpha(\phi) D_{x_n} v_i = D_{x_n}(D^\alpha(\phi) v_i) - D_{x_n}(D^\alpha(\phi)) v_i = 0, \tag{5}$$

for all $\alpha \in \mathcal{T}$. Thus any $D_{x_n} v_i$ is expressible as a $\mathcal{M}$-linear combination of the basis vectors $v_1, v_2, \ldots, v_k$. Since the basis is reduced and $(D_{x_n} v_i)_{\ell_j} = 0$ for $1 \le j \le k$, it follows that $D_{x_n}(v_i) = 0$. As this holds for any n and any i, it follows that the basis vectors $v_1, v_2, \ldots, v_k$ are contained in $\mathcal{N}_{\mathcal{T}} \cap \mathcal{K}^N$. □

(of Theorem 3.1). It only remains to show that the identical vanishing of the generalized Wronskians corresponding to Young-like sets is sufficient for linear dependence, whereas necessity was established in Section 2.

Let Y be the Young-like set satisfying Lemma 3.2. If $|Y| < N$, then one can adjoin appropriately chosen elements to Y to produce a larger Young-like set with cardinality exactly N. (For example, one may achieve this by using extra elements of the form $(k, 0, \ldots, 0)$.) So let B be a Young-like set with cardinality N that contains Y. By assumption, $W_B^\phi = 0$, so it follows that $\mathcal{N}_B$ is non-trivial. Since $\mathcal{N}_B \subseteq \mathcal{N}_Y = \mathcal{N}_\mathcal{T}$, it also holds that $\mathcal{N}_\mathcal{T}$ is non-trivial. By Lemma 3.3, this implies the existence of a non-trivial column vector $c = (c_1, c_2, \ldots, c_N)$ in $\mathcal{N}_\mathcal{T} \cap \mathcal{K}^N$. The column vector c satisfies $\phi c = 0$, which implies that the entries of ϕ are linear dependent over $\mathcal{K}$. □

We now show that the collection $\mathcal{Y}$ is the least, with respect to inclusion, among all collections of subsets of $\mathcal{T}$ that are sufficient for demonstrating linear dependence in a manner like Theorem 3.1.

Theorem 3.4. *Let $\mathcal{H}$ be any collection of subsets of $\mathcal{T}$ that would satisfy Theorem 2.1 with $\mathcal{G}$ replaced by $\mathcal{H}$ (or Theorem 3.1 with $\mathcal{Y}$ replaced by $\mathcal{H}$). Then $\mathcal{H}$ contains $\mathcal{Y}$.*

Proof. Let $A = \{\alpha^1, \alpha^2, \ldots, \alpha^N\}$ be an element of $\mathcal{Y}$, i.e. A is a finite Young-like set of cardinality N. Let $\phi = [x^{\alpha^1}, x^{\alpha^2}, \ldots, x^{\alpha^N}]$. The entries of ϕ are distinct monomials and thus linearly independent. So by hypothesis, $\mathcal{H}$ must contain a set B such that W_B^ϕ is not zero.

If $\beta = (b_1, b_2, \ldots, b_m) \notin A$, then for any $\alpha = (a_1, a_2, \ldots, a_N) \in A$, $D^\beta(x^\alpha) = 0$ as $b_j > a_j$ for some j, due to A being Young-like. Hence $D^\beta\phi = [0\ 0\ \ldots\ 0]$, if $\beta \notin A$. So W_B^ϕ is zero, if $B \neq A$. Therefore $A = B \in \mathcal{H}$. (In contrast, W_A^ϕ equals a non-zero constant, which may be verified by a careful calculation.)

□

It has already been noted that $\mathcal{Y} \subseteq \mathcal{G}$. In the case that $m = 1$ or $N \leq 2$, then $\mathcal{Y} = \mathcal{G}$. For $m \geq 2$ and $N \geq 3$, $\mathcal{Y}$ is a proper subcollection of $\mathcal{G}$. (For example, let $A = \{\alpha^1, \ldots, \alpha^N\}$, where $\alpha^j = (j-1, 0, \ldots, 0)$ for $1 \leq j \leq N-1$ and $\alpha^N = (0, N-1, 0, \ldots, 0)$, and note that $A \in \mathcal{G} \backslash \mathcal{Y}$.) Moreover, when $N \geq 3$ and $m \geq 2$, the cardinalities of $\mathcal{Y}$ and $\mathcal{G}$ will significantly differ, as the following will illustrate.

Young-like sets of dimension m and cardinality N have a correspondence with $(m-1)$-dimensional partitions of N. A method for calculating the number of k-dimensional partitions of N is known, as are the explicit results for $N \leq 6$ [1] (Section 11.4). This yields the following information about $|\mathcal{Y}|$.

(6)

N	$\lvert\mathcal{Y}\rvert$
2	m
3	$\frac{1}{2}m^2 + \frac{1}{2}m$
4	$\frac{1}{6}m^3 + m^2 - \frac{1}{6}m$
5	$\frac{1}{24}m^4 + \frac{3}{4}m^3 - \frac{1}{24}m^2 + \frac{1}{4}m$
6	$\frac{1}{120}m^5 + \frac{1}{3}m^4 + \frac{19}{24}m^3 - \frac{1}{3}m^2 + \frac{1}{5}m$

One can divide $\mathcal{G}$ into disjoint pieces $\mathcal{G}_{i_1,\dots,i_N} = \{A \in \mathcal{G} \mid |A \cap \mathcal{T}(j)| = i_j, \text{ for } 1 \leq j \leq N\}$ indexed by the increasing sequences $i_1 \leq i_2 \leq \cdots \leq i_N$ that satisfy $i_1 = 1$, $i_N = N$, and $i_j \geq j$ for $1 < j < N$. A calculation shows that $|\mathcal{T}(j) \backslash \mathcal{T}(j-1)| = \binom{m+j-2}{j-1}$ and consequentially that

$$|\mathcal{G}_{i_1,\dots,i_N}| = \prod_{j=2}^{N} \binom{\binom{m+j-2}{j-1}}{i_j - i_{j-1}}. \tag{7}$$

So (with $i_1 = 1$ and $i_N = N$)

$$|\mathcal{G}| = \sum_{i_2=2}^{N} \sum_{i_3=\max(i_2,3)}^{N} \cdots \sum_{i_{N-1}=\max(i_{N-2},N-1)}^{N} \prod_{j=2}^{N} \binom{\binom{m+j-2}{j-1}}{i_j - i_{j-1}}. \tag{8}$$

The results of calculating $|\mathcal{G}|$ for a few values of N are given below.

(9)

N	$\lvert\mathcal{G}\rvert$
2	m
3	$\frac{1}{2}m^3 + m^2 - \frac{1}{2}m$
4	$\frac{1}{12}m^6 + \frac{13}{24}m^5 + \frac{13}{12}m^4 + \frac{1}{8}m^3 - \frac{7}{6}m^2 + \frac{1}{3}m$
5	$\frac{m^{10}}{288} + \frac{29m^9}{576} + \frac{85m^8}{288} + \frac{253m^7}{288} + \frac{39m^6}{32} + \frac{29m^5}{576} - \frac{493m^4}{288} - \frac{35m^3}{48} + \frac{43m^2}{36} - \frac{m}{4}$

Calculating $|\mathcal{G}|$ can be cumbersome, particularly for large values of N. So it is worth noting that $|\mathcal{G}_{1,2,\dots,N}| = \prod_{j=2}^{N} \binom{m+j-2}{j-1}$ provides an simple lower bound for $|\mathcal{G}|$. (This term is dominant in (8) for N fixed and m large.)

In the case that $m = 2$, it holds that $|\mathcal{G}_{1,2,\dots,N}| = N!$ and $|\mathcal{Y}| = p(N)$, where $p(N)$ denotes the number of partitions of N. By the Hardy-Ramanujan formula, $p(N) \sim \frac{1}{4N\sqrt{3}} \exp\left[\pi\sqrt{\frac{2N}{3}}\right]$ as $N \to \infty$ [1] (Section 5.1).

As may be seen from the previous discussion, $|\mathcal{Y}|$ is substantially smaller than $|\mathcal{G}|$ in some various asymptotic ways. But even among small values of N and m the difference between $\mathcal{Y}$ and $\mathcal{G}$ can be appreciable, as the following table demonstrates.

(10)

$(\lvert\mathcal{Y}\rvert, \lvert\mathcal{G}\rvert)$	$m = 2$	$m = 3$	$m = 4$	$m = 5$
$N = 3$	$(3, 7)$	$(6, 21)$	$(10, 46)$	$(15, 85)$
$N = 4$	$(5, 37)$	$(13, 274)$	$(26, 1164)$	$(45, 3660)$
$N = 5$	$(7, 268)$	$(24, 5806)$	$(59, 55151)$	$(120, 334230)$

Remark: The arguments for proving Theorem 3.1 and Theorem 3.4 still apply with a more abstract definition of $\mathcal{O}$. For Theorem 3.1 and its supporting definitions we only required two essential properties of $\mathcal{O}$. One, we used that $\mathcal{O}$ is an integral domain to define the field $\mathcal{M}$. (In contrast, the ring of germs of C^∞ functions is not an integral domain.) Two, we required the presence of the derivations D_{x_j}, $1 \leq j \leq m$, on $\mathcal{O}$ satisfying $\cap_{j=1}^{m} \ker D_{x_j} = \mathcal{K}$ in $\mathcal{M}$. For the leastness theorem, Theorem 3.4, we used that $\mathcal{O}$ includes coordinate elements $x_1, x_2, \ldots, x_m$ dual to the derivations $D_{x_1}, D_{x_2}, \ldots, D_{x_m}$ and that $\mathcal{K}$ has characteristic zero. Both the definitions and results carry through in the case that $\mathcal{O}$ is a general $\mathcal{K}$-algebra with $\mathcal{K}$ and $\mathcal{O}$ satisfying these properties.

4. Some Further Remarks

The proof of Lemma 3.3 gives that the space of linear relations on the entries of ϕ over $\mathcal{K}$ is the $\mathcal{K}$-span of any reduced basis of $\mathcal{N}_{\mathcal{T}}$. So knowing $\mathcal{N}_{\mathcal{T}}$ offers more information than does only knowing whether the generalized Wronskians are zero. Furthermore the construction of Y in Lemma 3.2 offers an approach to calculating $\mathcal{N}_{\mathcal{T}}$, via $\mathcal{N}_Y$, which we present here in the form of an iterative algorithm. However we will operate with the range space, rather than the null space, of M_A^ϕ. So let $\mathcal{R}_A = \operatorname{span}_{\mathcal{M}}\{\phi_\alpha\}_{\alpha \in A}$.

Algorithm for Calculating $\mathcal{N}_Y$

(1) Initialize α to $(0, 0, \ldots, 0)$, A to $\emptyset$, and $\mathcal{R}_A$ to $\{0\}$.
(2) If $\phi_\alpha \notin \mathcal{R}_A$, then set A to $A \cup \{\alpha\}$ and $\mathcal{R}_A$ to $\operatorname{span}_{\mathcal{M}} \mathcal{R}_A \cup \{\phi_\alpha\}$.
(3) Set β to be the least element greater than α, according to the lexicographical ordering, such that $A \cup \{\beta\}$ is Young-like and has cardinality no greater than N. If no such β exists, skip to step 5.
(4) Set α to β and go back to step 2.
(5) Set Y to A and $\mathcal{N}_Y$ to $\mathcal{R}_A^\perp$. [Calculation complete.]

To conclude, we offer some casual thoughts that may be of interest to a computational point of view. (We are assuming that we have some means for addition, multiplication, inversion, and comparison to zero for the elements of $\mathcal{M}$ in stock, which may still be theoretical assumptions, depending on $\mathcal{M}$.)

For a practical implementation, one must select a data representation for the spaces $\mathcal{R}_A$ and $\mathcal{N}_Y$. Using a reduced $\mathcal{M}$-basis to represent these spaces seems to be a natural choice. For one, having the final product $\mathcal{N}_Y$ in reduced form is what we desire, as it yields a $\mathcal{K}$-basis for the space of linear relations. But also the linear algebra calculations in Step 2 and Step 5 may be very feasible to implement with a reduced basis representation. (If there should arise concerns about robustness, then exercising the freedom to choose the columns by which the basis is reduced might be beneficial.)

We also point out that this algorithm is adaptive. As specific multi-indices are either included or excluded from the set A in Step 2, certain subsequent multi-indices will be safely bypassed, by Step 3, on the grounds that their inclusion would prohibit the constructed multi-index set A from being Young-like. Consequentially, it may be interesting to see a rigorous analysis comparing the complexity of this algorithm with the complexity of computing all of the generalized Wronskians associated with Young-like sets.

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